

Exercise 1.2.2

Derive the heat equation for a rod assuming constant thermal properties and no sources.

- Consider the total thermal energy between x and $x + \Delta x$.
- Consider the total thermal energy between $x = a$ and $x = b$.

Solution

Part (a)

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters the left side of a shell at x , then that same amount must exit the right side of it at $x + \Delta x$ for the temperature to remain the same. If more (less) thermal energy enters at x than exits at $x + \Delta x$, then the amount of thermal energy in the rod will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

$$\text{rate of energy in} - \text{rate of energy out} = \text{rate of energy accumulation}$$

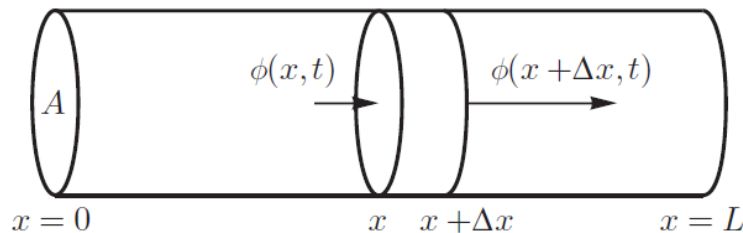


Figure 1: This is a schematic of the shell that the thermal energy flows through (differential formulation).

The flux is defined to be the rate that thermal energy flows through the shell per unit area, and we denote it by $\phi = \phi(x, t)$. If we let U represent the amount of energy in the shell, then the energy balance over it is

$$A\phi(x, t) - A\phi(x + \Delta x, t) = \left. \frac{dU}{dt} \right|_{\text{shell}}.$$

Factor $-A$ from the left side.

$$-A[\phi(x + \Delta x, t) - \phi(x, t)] = \left. \frac{dU}{dt} \right|_{\text{shell}}$$

The thermal energy in the shell is equal to its mass $\Delta m = \rho\Delta V = \rho A\Delta x$ times specific heat c times temperature $u(x, t)$.

$$-A[\phi(x + \Delta x, t) - \phi(x, t)] = \frac{\partial(\Delta m cu)}{\partial t}$$

Since the rod has constant properties, only u is a function of time.

$$-A[\phi(x + \Delta x, t) - \phi(x, t)] = \rho A\Delta x c \frac{\partial u}{\partial t}$$

Divide both sides by $A\Delta x$.

$$-\frac{\phi(x + \Delta x, t) - \phi(x, t)}{\Delta x} = \rho c \frac{\partial u}{\partial t}$$

Now take the limit as $\Delta x \rightarrow 0$.

$$\lim_{\Delta x \rightarrow 0} -\frac{\phi(x + \Delta x, t) - \phi(x, t)}{\Delta x} = \rho c \frac{\partial u}{\partial t}$$

The left side is how the negative of the first derivative of ϕ with respect to x is defined.

$$-\frac{\partial \phi}{\partial x} = \rho c \frac{\partial u}{\partial t}$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0 \frac{\partial u}{\partial x},$$

where K_0 is a proportionality constant known as the thermal conductivity. As a result, the energy balance becomes an equation solely for the temperature.

$$-\frac{\partial}{\partial x} \left(-K_0 \frac{\partial u}{\partial x} \right) = \rho c \frac{\partial u}{\partial t}$$

Therefore, the equation for the temperature is

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2},$$

or, dividing both sides by ρc and setting $k = K_0/\rho c$,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

Part (b)

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters the left side of a rod at $x = a$, then that same amount must exit the right side of it at $x = b$ for the temperature to remain the same. If more (less) thermal energy enters at $x = a$ than exits at $x = b$, then the amount of thermal energy in the rod will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

$$\text{rate of energy in} - \text{rate of energy out} = \text{rate of energy accumulation}$$

The flux is defined to be the rate that thermal energy flows through the rod per unit area, and we denote it by $\phi = \phi(x, t)$. If we let U represent the amount of energy in the rod, then the energy balance over it is

$$A\phi(a, t) - A\phi(b, t) = \frac{dU}{dt} \Big|_{\text{rod}}.$$

Factor $-A$ from the left side.

$$-A[\phi(b, t) - \phi(a, t)] = \frac{dU}{dt} \Big|_{\text{rod}}$$

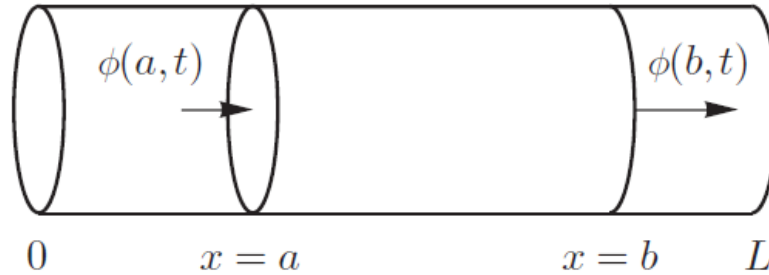


Figure 2: This is a schematic of the rod that the thermal energy flows through (integral formulation).

By the fundamental theorem of calculus, the term in square brackets is an integral.

$$-A \int_a^b \frac{\partial \phi}{\partial x} dx = \left. \frac{dU}{dt} \right|_{\text{rod}}$$

The thermal energy in the rod U is obtained by integrating the thermal energy density $e(x, t)$ over the rod's volume.

$$-A \int_a^b \frac{\partial \phi}{\partial x} dx = \frac{d}{dt} \int_{\text{rod}} e(x, t) dV$$

The volume differential is $dV = A dx$.

$$-A \int_a^b \frac{\partial \phi}{\partial x} dx = \frac{d}{dt} \int_a^b e(x, t) A dx$$

The thermal energy density is the density ρ times specific heat c times temperature $u(x, t)$.

$$-A \int_a^b \frac{\partial \phi}{\partial x} dx = \frac{d}{dt} \int_a^b \rho c u(x, t) A dx$$

Divide both sides by A and bring the minus sign and derivative inside the integrals.

$$\int_a^b \left(-\frac{\partial \phi}{\partial x} \right) dx = \int_a^b \rho c \frac{\partial u}{\partial t} dx$$

The integrands must be equal to one another.

$$-\frac{\partial \phi}{\partial x} = \rho c \frac{\partial u}{\partial t}$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0 \frac{\partial u}{\partial x},$$

where K_0 is a proportionality constant known as the thermal conductivity. As a result, the energy balance becomes an equation solely for the temperature.

$$-\frac{\partial}{\partial x} \left(-K_0 \frac{\partial u}{\partial x} \right) = \rho c \frac{\partial u}{\partial t}$$

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